Design of Linear Double Tapers in Rectangular Waveguides*

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Summary—This paper considers the problem of a taper connecting two uniform waveguides of arbitrary dimensions and propagating a single mode; an approximate expression for the reflection coefficient is derived. The special case of a linear double taper in rectangular waveguide is examined in detail for propagation in the TE_{10} mode. Approximate expressions for the reflection coefficient and voltage standing wave ratio as functions of the taper dimensions and free space wavelength are derived and experimentally verified.

Introduction

Tapered sections of waveguides are useful impedance matching devices with wide bandwidth characteristics. Of particular interest is the linear taper because of its ease of fabrication.

Several theoretical papers on reflections in nonuniform transmission lines are available in the literature. Matsumaru[1] recently analyzed linear and sinusoidal E-plane tapers in rectangular waveguides, supporting his theory with experimental evidence. He also pointed out that linear tapers perform almost as well as exponential tapers and better than shorter hyperbolic tapers. Quite often, however, it is necessary to design a double taper, *i.e.*, one that tapers simultaneously in both the E-plane and H-plane. Therefore, the microwave circuit engineer needs a method which enables him to design a linear taper which will match two uniform waveguides of arbitrary dimensions. This paper provides such a method for designing linear, single or double tapers in rectangular waveguide employing propagation in the TE_{10} mode. A similar analysis can be applied to other cases.

Reflection Coefficient of Tapered Waveguide

The exact theory of reflection in waveguides is quite complicated, particularly when the dimensions of the discontinuities are not small compared to a wavelength. For a small step discontinuity between guides of impedances Z_a and Z_b, the following well-known expression for the reflection coefficient can be used:[2]

$$\Gamma = \frac{Z_b - Z_a}{Z_b + Z_a}. \tag{1}$$

Consider a tapered waveguide of length L which connects two uniform waveguides of impedances Z_0 and Z_1 as shown in Fig. 1. Let x be the distance from the end of the taper where the impedance is Z_0 and assume that the impedance in the taper is a smooth function of x such that $Z(0) = Z_0$ and $Z(L) = Z_1$. Partition the interval $0 \leq x \leq L$ into N equal subintervals, and replace the taper with N uniform waveguides, each having a length of $\Delta x = L/N$. Let the impedance of the nth segment be $z_n = Z(x_n)$.

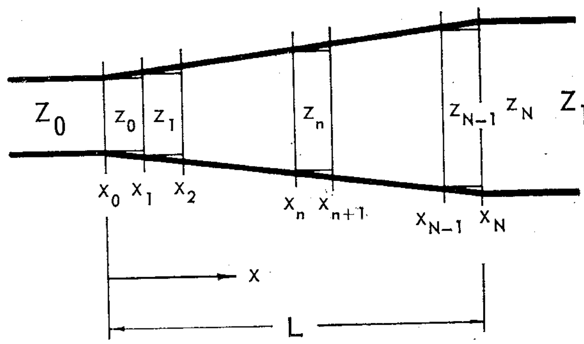

Fig. 1—Tapered waveguide of length L connecting uniform waveguides of impedances Z_0 and Z_1.

The total reflection coefficient is now a function of the N small reflections from the discontinuities at x_1, $x_2, \cdots, x_N$. If N is large, the reflection at each discontinuity becomes very small. Neglecting multiple reflections and the excitation of higher order modes, the N small reflections can be summed as complex vectors in the following manner;

$$\Gamma \approx \Gamma_1 \exp\left[-2\gamma_0\Delta x\right] + \Gamma_2 \exp\left[-2\gamma_0\Delta x - 2\gamma_1\Delta x\right] + \cdots + \Gamma_N \exp\left[-2\sum_{m=0}^{N-1}\gamma_m\Delta x\right],$$

or

$$\Gamma \approx \sum_{n=1}^{N}\Gamma_n \exp\left[-2\sum_{m=0}^{n-1}\gamma_m\Delta x\right], \tag{2}$$

where γ_n is the propagation constant in the nth segment and

$$\Gamma_n = \frac{z_n - z_{n-1}}{z_n + z_{n-1}}$$

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[1] K. Matsumaru, "Reflection coefficient of E-plane tapered waveguides," IRE Trans. on Microwave Theory and Techniques, vol. MTT-6, pp. 143–149; April, 1958.

[2] J. C. Slater, "Microwave Transmission," McGraw-Hill Book Co. Inc., New York, N. Y., p. 59; 1942.
S. A. Schelkunoff, "Electromagnetic Waves," D. Van Nostrand Co., Inc., New York, N. Y., pp. 319–320; 1943.

is the reflection coefficient at the nth discontinuity. Eq. (2) can be written as

$$\Gamma \approx \sum_{n=1}^{N} \frac{(z_n - z_{n-1})/\Delta x}{z_n + z_{n-1}} \exp\left[-2 \sum_{m=0}^{n-1} \gamma_m \Delta x\right] \Delta x.$$

Letting N approach infinity, the total reflection coefficient for the taper can be expressed by the integral

$$\Gamma = \int_0^L \frac{Z'(x)}{2Z(x)} \exp\left[-2 \int_0^x \gamma(\tau) d\tau\right] dx \qquad (3)$$

or

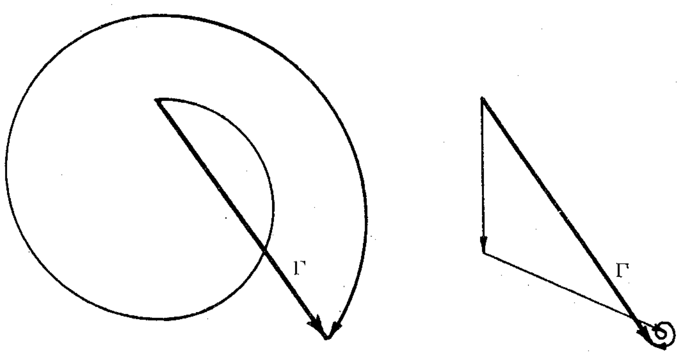
$$\Gamma = \int_0^L \frac{1}{2}\left(\frac{d}{dx} \ln Z\right) \exp\left[-2 \int_0^x \gamma d\tau\right] dx, \qquad (4)$$

where the prime denotes differentiation with respect to x.

The right side of (4) is a summation of infinitesimal vectors of slowly varying phase; Γ is equal to the resultant. In optics, a summation of this type is commonly called a vibration curve. A graphical representation of (4) is shown in Fig. 2(a). Integrating (4) by parts, the reflection coefficient becomes

$$\Gamma = \frac{1}{4\gamma_0}\left(\frac{d}{dx} \ln Z\right)_0 - \frac{1}{4\gamma_1}\left(\frac{d}{dx} \ln Z\right)_1 \exp\left[-2 \int_0^L \gamma dx\right] + \int_0^L \frac{d}{dx}\left(\frac{1}{4\gamma} \frac{d}{dx} \ln Z\right) \exp\left[-2 \int_0^x \gamma d\tau\right] dx. \qquad (5)$$

The right side of (5) is the sum of two vectors (terms 1 and 2) and a small vibration curve (term 3) somewhat similar in form to the vibration curve of (4). A graphical representation of (5) is shown in Fig. 2(b).



(a) (b)

Fig. 2—(a) Graphical representation of (4), (b) Graphical representation of (5).

Eqs. (4) and (5) are expressions for the reflection coefficient from a tapered section of transmission line or waveguide. The procedure from here depends upon the particular type of taper and waveguide under consideration. Eq. (5) yields a particularly simple result for an exponential taper in which γ is not a function of x. In this case the vibration curve term is zero and the reflection coefficient becomes

$$\Gamma = \frac{1}{4\gamma L} \ln \frac{Z_1}{Z_0} [1 - \exp(-2\gamma L)],$$

which is the same expression as that given by Ragan.[3]

Linear Taper in Rectangular Waveguide

The taper which will be examined in this paper is that of a linear double taper in rectangular waveguide employing the TE_{10} mode. Except for the phase factor, the integrand in the vibration curve (term 3) of (5) is the derivative of the product $1/2\gamma$ times the integrand of (4). Except where the frequency of operation is in the region near cutoff, γ and Z are slowly varying functions, and the vibration curve term in (5) is assumed to be negligible (See Appendix). This leaves the approximation,

$$\Gamma = \frac{1}{4\gamma_0}\left(\frac{d}{dx} \ln Z\right)_0 - \frac{1}{4\gamma_1}\left(\frac{d}{dx} \ln Z\right)_1 \exp\left[-2 \int_0^L \gamma dx\right], \qquad (6)$$

which is identical with the expression derived by Frank[4] using approximate solutions of the tapered transmission line equations. It should be noted that the logarithmic derivative is discontinuous at the ends of the taper; the values to be used are those just inside the tapered portion.

Fig. 3 illustrates the general configuration of a linear taper of length L connecting rectangular waveguides of impedances Z_0 and Z_1. In the taper section a and/or b are linear functions of x of the form

$$a = a(x) = a_0 + \frac{a_1 - a_0}{L} x$$

$$b = b(x) = b_0 + \frac{b_1 - b_0}{x} x.$$

To interpret (6) in terms of the TE_{10} mode in rectangular waveguide in free space dielectric, the integrated characteristic impedance defined on a voltage-current basis[5] is used. Thus, let

$$Z = \frac{\pi\eta_0}{2} \frac{b}{a\sqrt{1 - (\lambda/2a)^2}} \qquad (7)$$

and

$$\gamma = i\frac{2\pi}{\lambda_g} = i\frac{2\pi}{\lambda}\sqrt{1 - (\lambda/2a)^2}, \qquad (8)$$

where a and b are the width and height of the guide, respectively, λ is the free space wavelength, and λ_g is the guide wavelength. The logarithmic derivative in the taper is then found to be

[3] G. I. Ragan, "Microwave Transmission Circuits," Rad. Lab. Series, vol. 9, McGraw-Hill Book Co., Inc., New York, N. Y., p. 307; 1948.

[4] N. H. Frank, "Reflections from Sections of Tapered Transmission Lines and Wave Guides," Rad. Lab. Rep. No. 189; January 6, 1943.

[5] Schelkunoff, *op. cit.*, p. 319; Slater, *op. cit.*, pp. 183–185.

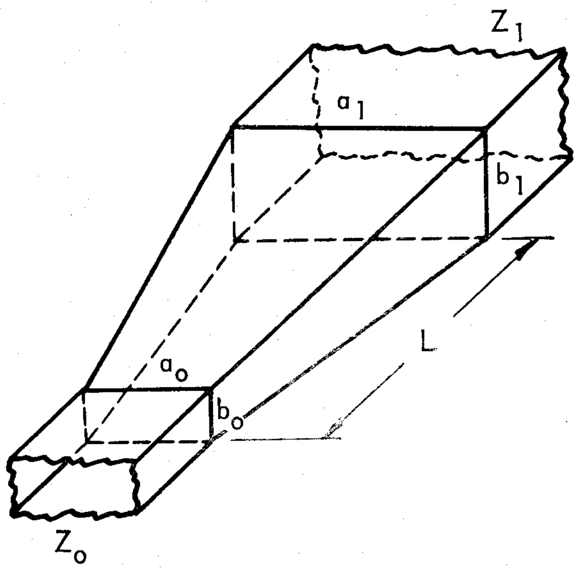


Fig. 3—Linear taper of length L connecting rectangular waveguides of impedances Z_0 and Z_1.

$$\frac{d}{dx}\ln Z = \frac{1}{L}\left[\frac{b_1 - b_0}{b} - \frac{(a_1 - a_0)/a}{1 - (\lambda/2a)^2}\right]. \tag{9}$$

The exponent in (6) can be written in the form

$$-2\int_0^L \gamma dx = -i4\pi\int_0^L \frac{dx}{\lambda_g}, \tag{10}$$

which is recognized as $-i4\pi$ times the number of guide wavelengths in the taper.

Substitution of (8), (9), and (10) into (6) yields the following expression for reflection coefficient,

$$\Gamma = \frac{i}{8\pi L/\lambda}\left[K_1 \exp(-i4\pi l) - K_0\right] \tag{11}$$

where

$$K_0 = \frac{(b_1 - b_0)/b_0 - [(a_1 - a_0)/a_0]/[1 - (\lambda/2a_0)^2]}{[1 - (\lambda/2a_0)^2]^{1/2}} \cdot \tag{12}$$

$$K_1 = \frac{(b_1 - b_0)/b_1 - [(a_1 - a_0)/a_1]/[1 - (\lambda/2a_1)^2]}{[1 - (\lambda/2a_1)^2]^{1/2}} \tag{13}$$

$$l = \int_0^L \frac{dx}{\lambda_g} = \frac{1}{\lambda}\int_0^L \sqrt{1 - (\lambda/2a)^2}\,dx. \tag{14}$$

Eq. (14) may be evaluated by using a series expansion of the radical and integrating term by term. The result is

$$l = \frac{L}{\lambda}\left\{1 + \sum_{n=1}^{\infty}(-1)^n \frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\cdots\left(\frac{-2n+3}{2}\right)}{n!(2n-1)} \cdot\left(\frac{\lambda}{2a_1 - 2a_0}\right)\left[\left(\frac{\lambda}{2a_0}\right)^{2n-1} - \left(\frac{\lambda}{2a_1}\right)^{2n-1}\right]\right\}. \tag{15}$$

Within the recommended operating range of standard waveguides, the first two terms of the series are sufficient; however, for frequencies near cutoff, additional terms should be used.

The absolute magnitude of the reflection coefficient is

$$|\Gamma| = \frac{1}{L/\lambda}\left[\frac{K_0^2 + K_1^2}{64\pi^2} - \frac{K_0K_1}{32\pi^2}\cos(4\pi l)\right]^{1/2}. \tag{16}$$

Using the relation

$$\text{VSWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|}, \tag{17}$$

the dominant mode voltage standing wave ratio (VSWR) can be calculated as a function of taper length and frequency for a linear taper connecting two specified waveguides. It is easy to calculate VSWR versus taper length for a fixed frequency since K_0 and K_1 are independent of length and l is proportional to length. To calculate VSWR versus frequency for a fixed taper length, the three constants must be evaluated for each frequency; however, the calculations are simple.

Linear E-Plane Taper in Rectangular Guide

In the special case of an E-plane taper an expression which is simpler than (16) can be derived for the absolute magnitude of the reflection coefficient. Since the guide width is constant, γ is not a function of x; therefore,

$$l = \int_0^L \frac{dx}{\lambda_g} = \frac{L}{\lambda_g}\cdot \tag{18}$$

Substituting (12), (13), and (18) into (11), the expression for the reflection coefficient becomes

$$\Gamma = \frac{i}{8\pi L/\lambda_g}\left[\frac{b_1 - b_0}{b_1}\exp(-4\pi L/\lambda_g) - \frac{b_1 - b_0}{b_0}\right]. \tag{19}$$

The absolute magnitude of the reflection coefficient is then

$$|\Gamma| = \frac{1}{8\pi L/\lambda_g}\left|1 - \frac{b_0}{b_1}\right|\left[1 + \left(\frac{b_1}{b_0}\right)^2 - 2\left(\frac{b_1}{b_0}\right)\cos(4\pi L/\lambda_g)\right]^{1/2}, \tag{20}$$

and the VSWR can be calculated from (17).

Eq. (19) is considerably different from the expression for reflection coefficient derived by Matsumaru;[6] however, the calculated VSWR's agree well for impedance ratios as great as $Z_1/Z_0 = 2$. With an impedance ratio of $Z_1/Z_0 = 2.8$ (20) predicts a slightly higher VSWR than that predicted by Matsumaru as illustrated in Fig. 4.

Eq. (20) was used to calculate the curve of VSWR vs taper length in Fig. 5 for an impedance ratio of

[6] Matsumaru, *op. cit.*, Eq. (8).

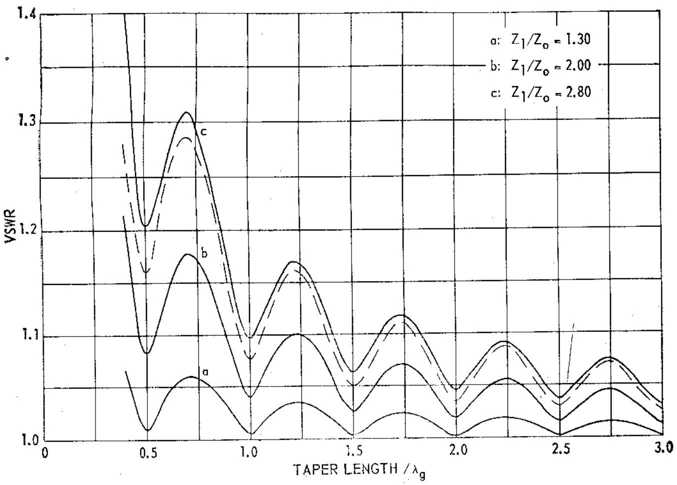


Fig. 4—VSWR of linear E-plane tapers. The solid curves were calculated from (20) and the dashed curve was calculated by Matsumaru.

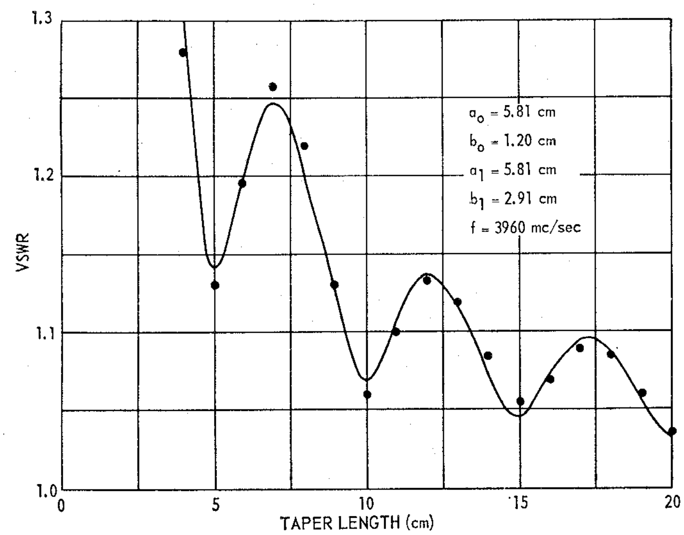


Fig. 5—VSWR versus taper length calculated from (20). The experimental points were reported by Matsumaru.

$Z_1/Z_0 = 2.42$. The experimental points were reported by Matsumaru[7] and used to verify his theoretical expression. It is seen that they can also serve to verify (20).

Experimental Investigation

In the experimental phase, a linear double taper was electroformed and tested over a wide band of frequencies. It was desirable to have the low end of the test frequency band near cutoff since the approximation of (6) does not hold there. A design frequency of 9500 mc was selected and the taper was fabricated to connect guides with dimensions of 0.900 inch × 0.400 inch and 0.750 inch × 0.600 inch; the latter has a cutoff frequency of 7869 mc. Based on (7) these waveguides have an impedance ratio of 2.33 at the design frequency. Measuring the VSWR from this taper gave an indication of

[7] *Ibid.*, Fig. 5.

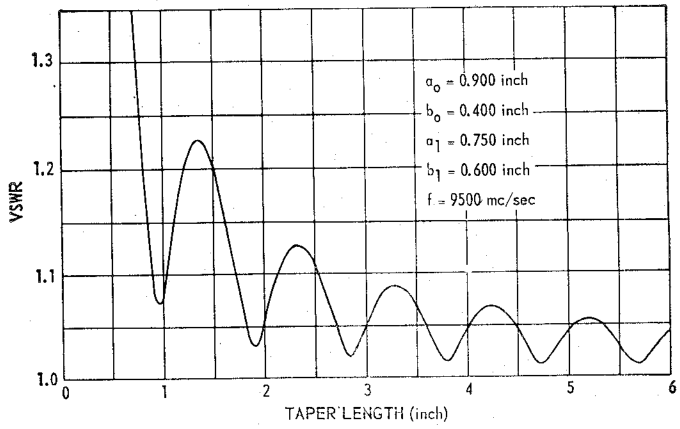


Fig. 6—Theoretical VSWR versus taper length for a linear double taper.

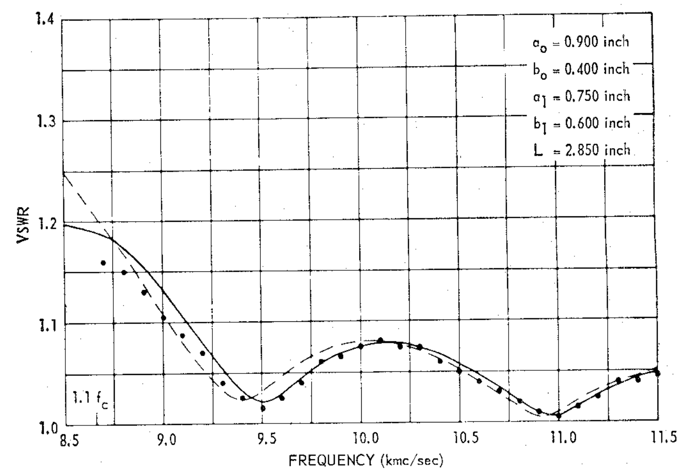


Fig. 7—VSWR versus frequency for a linear double taper. The curves were calculated from (16); the value of l was obtained from (15) using the first four terms of the series for the solid curve and the first two terms for the dashed curve. The points are experimental measurements.

the validity of (6) in the region near cutoff.

The curve of VSWR versus taper length shown in Fig. 6 was calculated using (16). The value of l was obtained from (15) using the first four terms of the series. A taper length of 2.850 inches was chosen, and curves of VSWR versus frequency were calculated from (16) for two values of l. Both curves are shown in Fig. 7.

The 0.750 inch × 0.600 inch guide was terminated by a sliding load, and the VSWR was measured at several frequencies with a standard X-band slotted line at the 0.900 inch × 0.400 inch end of the taper. The experimental points are also shown in Fig. 7. The agreement between the measured and theoretical VSWR is satisfactory at 10 per cent above cutoff; it becomes increasingly better at higher frequencies. Since the low end of the recommended operating bands for standard waveguides generally is 23 to 36 per cent above cutoff, (16) can be used to predict the VSWR from linear tapers for most practical applications.

Conclusion

The approximate expression for the absolute magnitude of the reflection coefficient in (16) enables the microwave circuit engineer to design linear, double or single tapers which will match rectangular waveguides of arbitrary dimensions employing the TE_{10} mode. The VSWR can be predicted as a function of the taper dimensions and the free space wavelength. In the experimental phase the measured VSWR agreed satisfactorily with the calculated value at 10 per cent above cutoff; the agreement became increasingly better for higher frequencies.

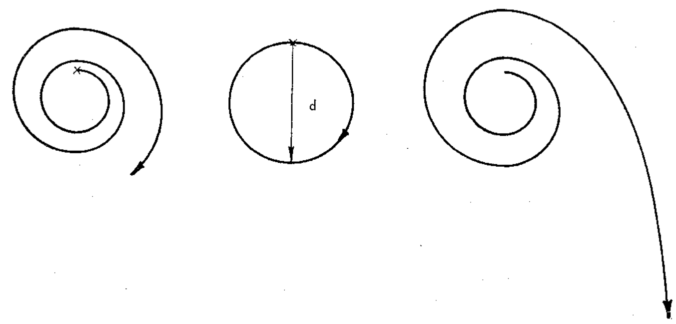


Fig. 8—Illustrations of (a) a typical vibration curve, (b) a circular vibration curve, and (c) a typical vibration curve from a taper having one end near cutoff.

Appendix

It is desirable to estimate the resultant of the vibration curve, term 3 of (5),

$$\int_0^L \frac{d}{dx}\left(\frac{1}{4\gamma}\frac{d}{dx}\ln Z\right)\exp\left[-2\int_0^x \gamma d\tau\right]dx.$$

The length T of the curve can be found by neglecting the phase factor in the integrand;

$$T = \left|\int_0^L \frac{d}{dx}\left(\frac{\lambda_g}{8\pi}\frac{d}{dx}\ln Z\right)dx\right|. \qquad (21)$$

Integrating and substituting from (12) and (13), the length of the vibration curve is found to be

$$T = \left|\frac{K_1 - K_0}{8\pi L/\lambda}\right|. \qquad (22)$$

The magnitude of the error in the approximation (6) must be less than T, and under normal circumstances, it will be considerably smaller than T since the vibration curve is a spiral through $4\pi l$ radians.

A better estimate of the magnitude of the error in (6) can be made by assuming a circular vibration curve of length T turning through $4\pi l$ radians as illustrated in Fig. 8(b). Then, the maximum magnitude of the error is approximately the diameter d of the circular path, where

$$d = \frac{T}{2\pi l} = \left|\frac{K_1 - K_0}{16\pi^2 l L/\lambda}\right|. \qquad (23)$$

For most applications, d is very small and (16) can be used with confidence. One must be cautious, however, at a frequency close to cutoff. If the width of one end of the taper is near the cutoff dimension, the phase shift per unit length will be very small at this end, and a vibration curve similar to that in Fig. 8(c) will result. In this case, the magnitude of the error may be larger than d but must still be smaller than T.

Acknowledgement

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